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# The rough calculus of treble defined integrals

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## Abstract

*In the present paper approximate formulare for triple integral that have great precision calculation are obtained. To obtain these formulare we used polynomials in interpolation to five nodes for simple defined integrals. By using the calculation mode of triple integrals, one can reduce the approximate calculation of triple integrals to double integrals by means of the approximate calculation of the simple integrals.*

**Keywords:** *interpolation*

## Introduction

In paper [2], the approximate calculation formulae for the simple definite integral

$$I = \int_a^b f(x)dx \tag{1}$$

where $f:[a,b] \to IR$ is a function integrable by $[a,b]$.

### The method of rectangles:

It is considered a unity of the distance of the interval $[a,b]$ with $n$ nodes:

$$\Delta : (x_0 = a < x_1 < x_2 < ... < x_i < x_{i+1} < ... < x_n = b)$$

with $h = \dfrac{b-a}{n}$ the distance of the unity, then we have the well-known formulae:

$$I_1 = h[f(x_0) + f(x_1) + .... + f(x_{n-1})] \tag{2}$$

$$I_2 = h[f(x_1) + f(x_2) + .... + f(x_n)] \tag{3}$$

The error of calculation for this method is:

$e_\tau \le A \cdot (b-a) \cdot h$, where $A = \sup\limits_{x\in[a,b]} |f'(x)|$, therefore $f \in C^1_{[a,b]}$.

## The trapeziums method

Considering the same unity as in the method of the rectangles, we have the formula:

$$I_3 = \frac{h}{2}\left[ f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i) \right] \tag{4}$$

which has the calculation error:

$e_\tau \le \frac{M \cdot (b-a)}{12} \cdot h^2$, where $M = \sup\limits_{x\in[a,b]} |f''(x)|$, therefore $f^2_{[a,b]}$.

## The Simpson method

It is considered the equidistant division with $2 \cdot n$ nodes

$\Delta : (x_0 = a < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{2n} = b)$

with $h = \frac{b-a}{2n}$ the distance of the division, then we have the formula:

$$I_4 = \frac{b-a}{6n}\left[ f(x_0) + f(x_{2n}) + 4\left( \sum_{k=0}^{n-1} f(x_{2k+1}) \right) + 2\left( \sum_{k=1}^{n-1} f(x_{2k}) \right) \right] \tag{5}$$

which has the calculation error:

$e_\tau \le \frac{(b-a)\cdot M}{180} \cdot h^4$, where $M = \sup\limits_{x\in[a,b]} \left| f^{(4)}_{(x)} \right|$, therefore $f \in C^4_{[a,b]}$.

## The method "A"

It is considered an equidistant division with 3*$n$ nodes

$$\Delta : (x_0 = a < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{3\cdot n} = b)$$

with $h = \frac{b-a}{3 \cdot n}$ the distance of the division, then we have the formula:

$$I_5 = \frac{3h}{8}\left[ f(a) + f(b) + 3\left( \sum_{k=0}^{n-1} f(x_{3k+1}) \right) + 3\left( \sum_{k=0}^{n-1} f(x_{3k+2}) \right) + 2\left( \sum_{k=0}^{n-1} f(x_{3k+3}) \right) \right] \tag{6}$$

which has the calculation error:

$e_\tau \le \frac{M \cdot (b-a)}{80} \cdot h^5$, where $M = \sup\limits_{x\in[a,b]} \left| f^{(4)}_{(x)} \right|$, therefore $f \in C^4_{[a,b]}$.

## The method "B"

It is considered an equidistant division with 4*$n$ nodes

$$\Delta : (x_0 = a < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{4\cdot n} = b)$$

with $h = \frac{b-a}{4n}$ the distance of the division, then we have the formula:

$$I_6 = \frac{2h}{45}\left[7(f(a)+f(b))+14\left(\sum_{k=1}^{n-1} f(x_{4k})\right)+32\left(\sum_{k=0}^{n-1} f(x_{4k+1})\right)+12\left(\sum_{k=0}^{n-1} f(x_{4k+2})\right)+\right.$$
$$\left.+32\left(\sum_{k=0}^{n-1} f(x_{4k+3})\right)\right] \tag{7}$$

which has the calculation error:

$e_\tau \le \frac{4 \cdot M \cdot (b-a)}{15 \cdot 21} \cdot h^6$, where $M = \sup_{x \in [a,b]} \left| f_{(x)}^{(5)} \right|$, there fore $f \in C_{[a,b]}^5$.

The most exact calculation method of the simple definite integral is the “B” method.

For the double definite integral

$$I = \iint_\Delta f(x,y)dxdy \tag{1'}$$

with $f : \Delta \to \mathbf{R}$ function on $\Delta$, where $\Delta \subset \mathbf{R}^2$ is limited by both of them: $y = y_1(x)$ and $y = y_2(x)$ with $y_1(x) \le y_2(x); (\forall) x \in [a,b]$, we can consider different divisions of the interval $[a,b]$, in accordance with the formulae $(2)-(7)$, and we have:

$$\begin{aligned} I &= \iint_D f(x,y)dxdy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)dy\right]dx = \\ &= \sum_{i=0}^{p-1} \int_{x_i}^{x_{i+1}} \left[\int_{y_1(x_i)}^{y_2(x_i)} (x,y)dy\right]dx = \sum_{i=0}^{p-1} \int_{x_i}^{x_{i+1}} \varphi(x)dx \end{aligned} \tag{8}$$

where:

$p$ could be: $n$, $2 \cdot n$, $3 \cdot n$ or $4 \cdot n$, and

$$\varphi(x) = \int_{y_1(x)}^{y_2(x)} f(x,y)dy \tag{9}$$

For the calculation of the simple integrals:

$$J_i = \int_{x_i}^{x_{i+1}} \varphi(x)dx \tag{10}$$

and

$$\varphi(x_i) = \int_{y_1(x_i)}^{y_2(x_i)} f(x_i, y)dy \tag{11}$$

we can use the formulae $(2)-(7)$, and we will obtain 36 calculation methods of double integrals.

The most exact method is that in which the integrals (10) and (11) are calculated by mean of the “B” method also called the method “B×B”, which has the formulae:

$$I_6 \overset{(7)}{=} \frac{2 \cdot h}{45}\left[7(\varphi(x_0)+\varphi(x_{4\cdot n}))+14\left(\sum_{k=1}^{n-1}\varphi(x_{4\cdot k})\right)+32\left(\sum_{k=0}^{n-1}\varphi(k_{4k+1})\right)+\right.$$
$$\left.+12\left(\sum_{k=0}^{n-1}\varphi(x_{4k+2})\right)+32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+3})\right)\right] \tag{12}$$

and

$$\varphi(x_i) \overset{(7)}{=} \frac{2h_1}{45}\left[7(f(x_i,y_0)+f(x_i,y_{4\cdot m}))+14\left(\sum_{k=1}^{m-1}f(x_i,y_{4\cdot k})\right)+\right.$$
$$\left.+32\left(\sum_{k=0}^{m-1}f(x_i,y_{4k+1})\right)+12\left(\sum_{k=0}^{m-1}f(x_i,y_{4k+2})\right)+32\left(\sum_{k=0}^{m-1}f(x_i,y_{4k+3})\right)\right] \tag{13}$$

where the interval [a,b] has the equidistand division $\Delta_1$ with the norm $\nu(\Delta)=h=\frac{b-\alpha}{4n}$, and the intervals $[y_1(x_i), y_2(x_i)]$ have the equidistant divisions $\Delta_i^{'}$ with the norms $\nu(\Delta_i^{'})=h_1=\frac{y_2(x_i)-y_1(x_i)}{4\cdot m}$, for $i\in[0,1,...,4\cdot m]$.

## The method "B×B×B"

We aim at the approximate calculation of the definite triple integral I, where:

$$I \ = \iiint_V f(x,y,z)dxdydz \tag{14}$$

with $f:V\to\mathbf{R}$ integrable on V and $V\subset\mathbf{R}^3$ and we presuppose that the domain V is limited by two surfaces: $z=z_1(x,y)$ and $z=z_2(x,y)$ with $z_1(x,y)\le z_2(x,y)$ for $(\forall)(x,y)\in D$, where $D\subset\mathbf{R}^2$ represents the projection of the V corp on the plan xoy, and the domain $D: y=y_1(x)$ and $y=y_2(x)$, is limited by $y_1(x)\le y_2(x)$ for $x\in[a,b]$ where [a,b] is the projection of D on the axis Ox.

From the calculation mode of the triple integral we have:

$$I=\iiint_V f(x.y.z)dxdydz=\iint_D\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dxdy \tag{15}$$

If we note

$$g(x,y)=\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \tag{16}$$

then (15) becomes:

$$I=\iint_D g(x,y)dxdy \tag{15'}$$

For the approximate calculation of the integral (15') 36 calculation methods of calculation of double integrals can be used, and for the calculation of the simple integral (16), 6 more methods can be used. In conclusion, for the approximate calculation of the definite triple integrals 216 methods can be used. The most exact method of all these methods is that which is used for the double integral (15') is used the method ''B×B", and for the simple integral (16) the „B"method is used.

This method will be called the method "B×B×B".

Using the relations (12) and (13) for the double integral (15') it results that the approximate value of the integral (14) is given by the formulae:

$$I = \frac{2 \cdot h}{45}\left[7(\varphi(x_0) + \varphi(x_{4\cdot n})) + 14\left(\sum_{k=1}^{n-1}\varphi(x_{4\cdot k})\right) + 32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+1})\right) + \right.$$

$$\left. + 12\left(\sum_{k=0}^{n-1}\varphi(x_{4k+2})\right) + 32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+3})\right)\right] \tag{17}$$

where the interval [a,b] has the equidistant $\Delta_1$, with $4 \cdot n$ nodes and the norm $\nu(\Delta_1) = \dfrac{b-a}{4 \cdot n} = h$, and

$$\begin{aligned} \varphi(x_i) = \frac{2 \cdot h_1}{45}&\left[7(g(x_i, y_0) + g(x_i, y_{4\cdot m})) + 14\left(\sum_{k=1}^{m-1} g(x_i, y_{4\cdot k})\right) + \right. \\ &\left. + 32\left(\sum_{k=0}^{m-1} g(x_i, y_{4k+1})\right) + 12\left(\sum_{k=0}^{m-1} g(x_i, y_{4k+2})\right) + 32\left(\sum_{k=0}^{m-1} g(x_i, y_{4k+3})\right)\right] \end{aligned} \tag{18}$$

where the interval $[y_1(x_i), y_2(x_i)]$ has the division equidistant $\Delta_2$ with $4 \cdot m$ nodes and the norm

$$\nu(\Delta_2) = \frac{y_2(x_i) - y_1(x_i)}{4 \cdot m} = h_1,$$

and

$$\begin{aligned} g(x_i, y_i) = \frac{2h_2}{45}&\left[7\left(f(x_i, y_j, z_0) + f(x_i, y_j, z_{4\cdot p})\right) + \right. \\ &+ 14\left(\sum_{k=1}^{p-1} f(x_i, y_j, z_{4\cdot k})\right) + 32\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+1})\right) + \\ &\left. + 12\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+2})\right) + 32\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+3})\right)\right. \end{aligned} \tag{19}$$

for $i \in [0,1,...4 \cdot n]$, $j \in [0,1,2,...4 \cdot m]$, where the interval $[z_1(x_i y_j), z_2(x_i, y_j)]$ has the division equidistant $\Delta_3$ with $4 \cdot p$ nodes and the norm

$$\nu(\Delta_3) = \frac{z_2(x_i, y_j) - z_1(x_i, y_j)}{4 \cdot p} = h_2.$$

The formulae (17), (18) and (19) represent the method "B×B×B" which has the calculation error:

$$e_\tau \le \frac{4 \cdot M \cdot (b-a)}{15 \cdot 21} \cdot h^6 \tag{20}$$

where

$$M = \sup_{(x,y,z)\in V} \left| \frac{\nu^5 f}{\nu x^5}(x, y, z) \right|, \text{ if } f \in C^5_{[a,b]} \text{ as against x.}$$

## References


1. D i n u ,  T .  - *Analiză numerică,* Editura U.P.G. din Ploieşti, 2000
2. D i n u ,  T .  - Calculul aproximativ al integralelor simple*, Buletinul U.P.G. din Ploieşti*, nr.1/2004
3. D i n u ,  T .  - Calculul aproximativ al integralelor duble*, Buletinul U.P.G. din Ploiesti*, nr.1/2005
4. G r i g o r e ,  G h . - *Lecţii de Analiză numerică,* Facultatea de Matematică din Bucureşti (curs litografiat)
5. M a r i n e s c u ,  G h . - *Analiză numerică,* Editura Academiei Romane, Bucureşti, 1974


# Calculul aproximativ al integralelor triple

## Rezumat

*În lucrare sunt obținute formule de calcul aproximativ pentru integralele triple care au precizie de calcul foarte mare. Pentru obținerea acestor formule au fost folosite polinoame de interpolare cu cinci noduri pentru integrala definită simplă. Utilizînd modul de calcul al integralelor triple, se reduce calculul aproximativ al integralelor triple la calculul aproximativ al integralelor duble folosind calculul aproximativ al integralelor simple.*